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# A Contribution to the Rainich Theory of the Neutrino Field

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*Abstract*

The neutrino-gravitational field is divided into fourteen distinct classes. This classification is shown also to determine the class of the space-time. Then for four particular classes of space-time, necessary and sufficient conditions on the concomitants of the Ricci tensor are given for it to admit a neutrino field. It is further shown that for three of these classes the neutrino field is determined uniquely by the metric of space-time.

## 1. *Introduction*

Wheeler (1960) has pointed out that the outstanding problem in geometrodynamics is to find out whether neutrino fields are included in that theory. In the same way as in the Rainich theory for electromagnetic fields what is required is to find necessary and sufficient conditions on the Ricci tensor for it to admit a neutrino field. Then given the metric tensor, and hence the Ricci tensor, we need to show that the neutrino field is defined uniquely. This scheme has still not been established. In fact the energy momentum tensor of the neutrino field is of such a form that a complete Rainich-type theory would be extremely complicated. In this paper we present a restricted theory. We first give a suitable classification of the neutrino gravitational field and then give a complete Rainich-type theory involving concomitants of the Ricci tensor for a number of particular classes.

The notations used in this paper will be the same as those used in Griffiths & Newing (1971a).

## 2. *Classification*

Plebanski (1964) has obtained a classification of the trace-free energy momentum tensor according to its eigenvalues and eigenvectors. This is

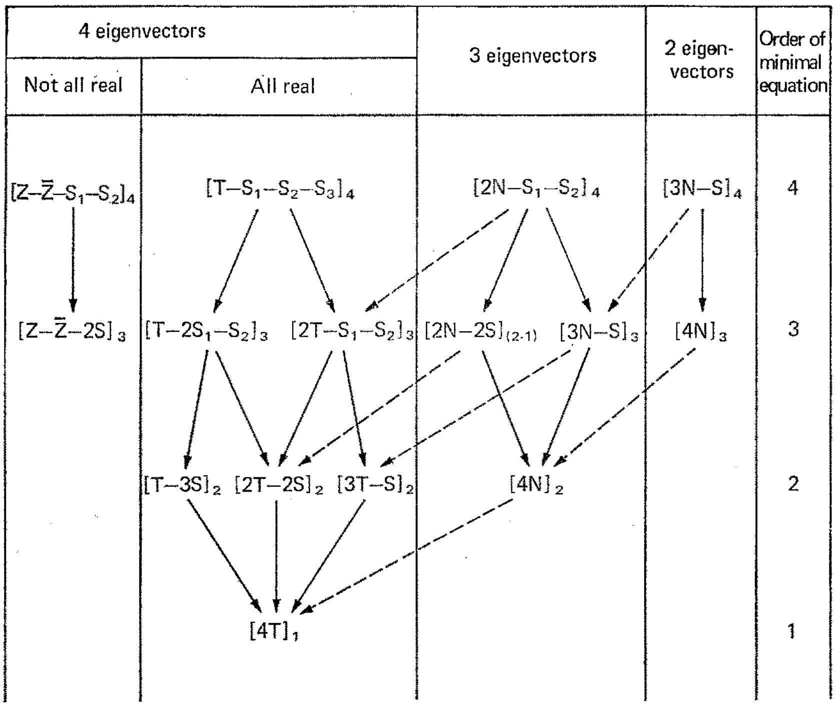


Figure 1.—Plebanski classification of the trace-less energy momentum tensor. This is given in Plebanski's notation. The eigenvalues are described inside the square bracket. The symbol $Z$ is used to represent a complex eigenvalue. The symbol $T$, $N$ or $S$ is used to represent an eigenvalue with which is associated respectively a time-like eigenvector, no time-like eigenvector but a null eigenvector, or only space-like eigenvectors. The number sometimes placed before these symbols denotes the number of times an eigenvalue is repeated if more than once. The numbers outside the square brackets are the indices of nil-potency in the same order in which the eigenvalues are given. Where there is no ambiguity these are replaced by the sum of the indices of nil potency. The arrows indicate degenerations. Here we have used full arrows to denote degenerations caused by two distinct eigenvalues becoming the same and dotted arrows to denote degenerations caused by a decrease in the order of the minimal equation.

given in Fig. 1. Since the trace of the neutrino energy momentum tensor is zero, Plebanski's method may be used immediately to classify the neutrino field. It can be seen from Fig. 1 that Plebanski has distinguished fifteen classes. However the class $[4T]_1$ is the case of empty space. Neutrino fields may be contained in the remaining fourteen classes. This has been shown by Griffiths (1971).

Einstein's field equations for neutrino gravitational fields may be written in suitable units as

$$R_{\mu\nu} = -E_{\mu\nu} \tag{2.1}$$

since the trace of the neutrino energy momentum tensor is zero. The first Rainich-type condition on the Ricci tensor for it to admit a neutrino field is therefore

$$R_\alpha{}^\alpha = 0$$

In the following work this condition will be assumed to be satisfied.

Now since the Ricci tensor is trace-free it may be classified according to Plebanski's method. The class of the Ricci tensor will then determine the class of the neutrino field. According to the approach of geometrodynamics we must start with the Ricci tensor. We can immediately tell to which Plebanski class this belongs. Then depending on the particular class we may be able to find further conditions on the Ricci tensor for it to admit a neutrino field. In the next section we show how this may be done for four particular classes giving conditions which are both necessary and sufficient.

## 3. *Conditions for the Neutrino Field of Some Classes*

We will only consider fields belonging to the four classes $[2N - S_1 - S_2]_4$, $[2N - 2S]_{(2-1)}$, $[3N - S]_3$ and $[4N]_2$. If the Ricci tensor is of one of these classes it will possess three eigenvectors. There will be a null eigenvector $L_\mu$ corresponding to the eigenvalue $N$ and two space-like eigenvectors $X_\mu$ and $Y_\mu$ corresponding to the eigenvalues $S_1$ and $S_2$ respectively. The eigenvectors are assumed to be normalised so that

$$L_\alpha L^\alpha = 0, \qquad X_\alpha X^\alpha = -1, \qquad Y_\alpha Y^\alpha = -1$$
$$L_\alpha X^\alpha = 0, \qquad L_\alpha Y^\alpha = 0, \qquad X_\alpha Y^\alpha = 0$$

From these three vectors it is possible to build up a null tetrad $L_\mu$, $N_\mu$, $M_\mu$, $\bar{M}_\mu$ where $N_\mu$ is defined such that $L_\alpha N^\alpha = 1$, $N_\alpha X^\alpha = 0$ and $N_\alpha Y^\alpha = 0$, and $M_\mu = (1/\sqrt{2})\,(X_\mu - iY_\mu)$ and $\bar{M}_\mu = (1/\sqrt{2})\,(X_\mu + iY_\mu)$. $M_\mu$ and $\bar{M}_\mu$ are chosen so that the tensor $(L_\mu M_\nu - M_\mu L_\nu)$ is self-dual. Such a tetrad is defined uniquely just from a consideration of the Ricci tensor. In the cases where $S_1 = S_2$ the space-like eigenvectors are not defined uniquely, but this lack of uniqueness is not significant.

The Ricci tensor for the classes considered may be expressed in the form

$$R_{\mu\nu} = aL_\mu L_\nu + N(L_\mu N_\nu + N_\mu L_\nu) - S_1 X_\mu X_\nu - S_2 Y_\mu Y_\nu, \qquad a \neq 0$$

Since the Ricci tensor is trace-free the sum of the eigenvalues will be zero, i.e.

$$2N + S_1 + S_2 = 0$$

Now the energy momentum tensor of the neutrino field for these classes may be put in the form

$$E_{\mu\nu} = 2i(\bar{\gamma}-\gamma)\,l_\mu l_\nu + i(\alpha-\bar{\beta}-\bar{\tau})\,(l_\mu m_\nu + m_\mu l_\nu)$$
$$-i(\bar{\alpha}-\beta-\tau)\,(l_\mu \bar{m}_\nu + \bar{m}_\mu l_\nu) + 2\omega(l_\mu n_\nu + n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu)$$
$$+\,2i\bar{\sigma} m_\mu m_\nu - 2i\sigma \bar{m}_\mu \bar{m}_\nu$$

where $l_\mu$, $n_\mu$, $m_\mu$, $\bar{m}_\mu$ is the neutrino tetrad constructed about the neutrino flux vector $l_\mu$, which defines a null geodesic congruence ($\kappa = 0$). The Greek letters $\rho$ ($= \theta + i\omega$), $\sigma$, $\tau$, $\kappa$, $\alpha$, $\beta$, $\gamma$, $\epsilon$ denote spin coefficients associated with the neutrino tetrad. For the two classes $[2N - S_1 - S_2]_4$ and $[2N - 2S]_{(2-1)}$ it is always possible to make a $\psi$-transformation ($m_\mu \to m'_\mu = m_\mu + \psi l_\mu$) on the neutrino tetrad that will make the coefficient $(\alpha - \bar{\beta} - \bar{\tau})$ zero. For a field to belong to the remaining two classes $[3N - S]_3$ and $[4N]_2$ the coefficient $(\alpha - \bar{\beta} - \bar{\tau})$ must be zero anyway. With this coefficient zero the gravitational field equations show that the neutrino tetrad may be related to the eigenvectors of the Ricci tensor by

$$l_\mu = \lambda L_\mu, \qquad m_\mu = e^{i\phi} M_\mu, \qquad \bar{m}_\mu = e^{-i\phi} \bar{M}_\mu$$

i.e.

$$L_\mu = \lambda^{-1} l_\mu, \quad X_\mu = \frac{1}{\sqrt{2}}(e^{-i\phi} m_\mu + e^{i\phi} \bar{m}_\mu), \quad Y_\mu = \frac{i}{\sqrt{2}}(e^{-i\phi} m_\mu - e^{i\phi} \bar{m}_\mu)$$

The neutrino energy momentum tensor may now be expanded in the form

$$E_{\mu\nu} = 2i(\bar{\gamma}-\gamma)\,\lambda^2 L_\mu L_\nu + 2\omega(L_\mu N_\nu + N_\mu L_\nu)$$
$$+\,(2\omega + i\bar{\sigma} e^{2i\phi} - i\sigma e^{-2i\phi})\,X_\mu X_\nu + (2\omega - i\bar{\sigma} e^{2i\phi} + i\sigma e^{-2i\phi})\,Y_\mu Y_\nu$$
$$+\,(\bar{\sigma} e^{2i\phi} + \sigma e^{-2i\phi})\,(X_\mu Y_\nu + Y_\mu X_\nu)$$

From this it may be seen that Einstein's field equations (2.1) give us the conditions

$$a = 2i(\gamma - \bar{\gamma})\,\lambda^2$$

or

$$\frac{ia}{2\lambda^2} = (\bar{\gamma} - \gamma) = m_{\alpha;\beta} n^\beta \bar{m}^\alpha \tag{3.1}$$

$$N = -2\omega = i l_{\alpha;\beta}(\bar{m}^\beta m^\alpha - m^\beta \bar{m}^\alpha) \tag{3.2}$$

$$\bar{\sigma} e^{2i\phi} + \sigma e^{-2i\phi} = 0$$

or

$$\sigma = \pm i|\sigma|\, e^{2i\phi} = l_{\alpha;\beta} m^\alpha m^\beta \tag{3.3}$$

and using this

$$S_1 = 2\omega \pm 2|\sigma| \tag{3.4}$$

$$S_2 = 2\omega \mp 2|\sigma| \tag{3.5}$$

In addition to these equations we also have the conditions

$$\kappa = 0 = l_{\alpha;\beta} l^{\beta} m^{\alpha} \tag{3.6}$$

and

$$\alpha - \bar{\beta} - \bar{\tau} = 0 = -(m_{\alpha;\beta} \bar{m}^{\beta} + l_{\alpha;\beta} n^{\beta}) \bar{m}^{\alpha} \tag{3.7}$$

We also require that the neutrino equations be satisfied. These may be given in terms of the neutrino tetrad by

$$(l_{\mu} m^{\alpha} - m_{\mu} l^{\alpha})_{;\alpha} = m^{\alpha} l_{\alpha;\mu} \tag{3.8}$$

It may be of interest to note that the conditions (3.2) and (3.6) may be replaced by the single condition on the flux vector

$$\sqrt{(-g)}\, \epsilon_{\mu\nu\alpha\beta} l^{\nu} l^{\alpha;\beta} = N l_{\mu}$$

Also conditions (3.4) and (3.5) imply that

$$l_{(\alpha;\beta)} l^{\alpha;\beta} l_{\mu} l_{\nu} - l_{\mu;\alpha} l_{\nu;\beta} l^{\alpha} l^{\beta} = \tfrac{1}{8}(S_1 - S_2)^2 l_{\mu} l_{\nu}$$

as a consequence of (3.6) and (3.8). These may be given in terms of the null eigenvector by

$$\sqrt{(-g)}\, \epsilon_{\mu\nu\alpha\beta} L^{\nu} L^{\alpha;\beta} = \lambda^{-1} N L_{\mu}$$
$$L_{(\alpha;\beta)} L^{\alpha;\beta} L_{\mu} L_{\nu} - L_{\mu;\alpha} L_{\nu;\beta} L^{\alpha} L^{\beta} = \tfrac{1}{8}\lambda^{-2}(S_1 - S_2)^2 L_{\mu} L_{\nu}$$

Purely from a consideration of the Ricci tensor the eigenvalues $N$, $S_1$ and $S_2$ and the tetrad $L_{\mu}$, $N_{\mu}$, $M_{\mu}$, $\bar{M}_{\mu}$ (and hence its associated spin coefficients) are determined uniquely. The field equations (3.1–3.8) may easily be rewritten in terms of these known quantities and the two scalars $\lambda$ and $\phi$. These are now given in the same order. For convenience we denote the spin coefficients of the known tetrad by the suffix '0'.

$$\phi_{,\alpha} N^{\alpha} = i(\gamma_0 - \bar{\gamma}_0) - \frac{a}{2\lambda} \tag{3.9}$$

$$N = -2\lambda\omega_0 \tag{3.10}$$

$$\arg \sigma_0 = \pm \frac{\pi}{2} \tag{3.11}$$

$$|\sigma_0| = \pm \frac{1}{4\lambda}(S_1 - S_2) \tag{3.12}$$

$$R_{\alpha}{}^{\alpha} = 0 \tag{3.13}$$

$$\kappa_0 = 0 \tag{3.14}$$

$$\phi_{,\alpha} \bar{M}^{\alpha} = i(\alpha_0 - \bar{\beta}_0 - \bar{\tau}_0) \tag{3.15}$$

$$\left(\frac{\lambda_{,\alpha}}{\lambda} + i\phi_{,\alpha}\right) L^{\alpha} = 2(\rho_0 - \epsilon_0) \tag{3.16}$$

$$\left(\frac{\lambda_{,\alpha}}{\lambda} + i\phi_{,\alpha}\right) M^{\alpha} = 2(\tau_0 - \beta_0) \tag{3.17}$$

If these equations may be satisfied for some $\lambda$ and $\phi$ then the space-time will admit a neutrino field. These equations form both necessary and sufficient conditions for a space-time of the Plebanski classes $[2N-S_1-S_2]_4$, $[2N-2S]_{(2-1)}$, $[3N-S]_3$ and $[4N]_2$ to admit a neutrino field. The argument may be expressed in the following way.

If the Ricci tensor of a space-time satisfies (3.13) and belongs to one of the four Plebanski classes mentioned above, a tetrad may be obtained as described. If this tetrad now satisfies (3.11) and (3.14), i.e.

$$L_{\alpha;\beta}(M^\alpha M^\beta + \bar{M}^\alpha \bar{M}^\beta) = 0, \qquad L_{\alpha;\beta} L^\beta M^\alpha = 0$$

(These are the eigenvector conditions $L_{\alpha;\beta}(X^\alpha X^\beta - Y^\alpha Y^\beta) = 0$, $L_{\alpha;\beta} X^\alpha L^\beta = 0$, $L_{\alpha;\beta} Y^\alpha L^\beta = 0$); and if the definition of $\lambda$ in (3.10) and (3.12) is consistent and satisfies the requirements of (3.16) and (3.17) that

$$\lambda_{,\alpha} L^\alpha = \lambda(\rho_0 + \bar{\rho}_0 - \epsilon_0 - \bar{\epsilon}_0)$$

and

$$\lambda_{,\alpha} M^\alpha = \lambda(3\tau_0 - \beta_0 - \bar{\alpha}_0)$$

and if a $\phi$ can be found to satisfy (3.9), (3.15), (3.16) and (3.17), that is if the equation

$$\phi_{,\mu} = \left[i(\gamma_0 - \bar{\gamma}_0) - \frac{a}{2\lambda}\right] L_\mu + [2\omega_0 + i(\epsilon_0 - \bar{\epsilon}_0)] N_\mu - i(\alpha_0 - \bar{\beta}_0 - \bar{\tau}_0) M_\mu + i(\bar{\alpha}_0 - \beta_0 - \tau_0) \bar{M}_\mu$$

is integrable; then the space-time admits a neutrino field.

This is a restricted Rainich-type theory in the sense that conditions on the metric are found for it to admit a neutrino field of some classes. However, these conditions are given in terms of the concomitants of the Ricci tensor and not the Ricci tensor itself.

## 4. *Uniqueness*

If a space-time of the Plebanski classes $[2N-S_1-S_2]_4$, $[2N-2S]_{(2-1)}$ or $[3N-S]_3$ admits a neutrino field then the neutrino field is determined by the metric. However, neutrino gravitational fields of the class $[4N]_2$ are not always uniquely determined.

These statements may be verified by considering the theorems given by Griffiths & Newing (1971b). Since fields of these classes possess only one null eigenvector, this must be proportional to the neutrino flux vector. In this way the direction of the flux vector, is defined and for this case a non-unique field must have an energy momentum tensor of the form

$$E_{\mu\nu} = A l_\mu l_\nu + B(l_\mu m_\nu + m_\mu l_\nu) + \bar{B}(l_\mu \bar{m}_\nu + \bar{m}_\mu l_\nu)$$

If $B$ is non-zero this does not belong to the classes being considered. Therefore neutrino gravitational fields of the classes $[2N-S_1-S_2]_4$,

$[2N-2S]_{(2-1)}$ and $[3N-S]_3$ must be uniquely defined by the metric. In the case where $B=0$ we have pure radiation fields of the class $[4N]_2$ which admits an arbitrariness as indicated by Griffiths & Newing (1971b).

## 5. *Conclusion*

We have presented here a complete Rainich-type theory of neutrino gravitational fields for particular classes of the Ricci tensor. It may be possible at a later date to extend this work to cover some of the remaining classes as well. However the classes considered here include the classes which have the greatest physical interest. The class $[4N]_2$ is that known as the neutrino pure radiation field which has stimulated the greatest interest so far. Audretsch (1971) has shown that this class and $[2N-2S]_{(2-1)}$ are the two which contain neutrino fields with causal behaviour. All of these classes also contain neutrino fields with positive energy density.

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